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EINSTEIN-YANG-MILLS EXTENSIONS OF INDUCED MATTER THEORY

Hossein Abolghasem[1]†, Andrew P. Billyard[2]‡, Alan A. Coley[3]† and Des McManus*

† *Dept. of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Canada, B3H 3J5*

‡ *Dept. of Physics, Dalhousie University, Halifax, Canada, B3H 3J5*

* *Bank of Canada, Ottawa, Canada, K1A 0G9*

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In induced matter theory, Einstein's four-dimensional theory with matter is normally embedded in higher-dimensional vacuum general relativity; as a result, the properties of matter have a geometrical origin. In this paper we investigate Einstein-Yang-Mills extensions of induced matter theory, in which the higher-dimensional vacuum field equations of Einstein-Yang-Mills type reduce to the field equations of Einstein's four-dimensional theory with matter. Such extensions allow much richer forms for the induced matter than in the usual theory. We study isotropic and spatially homogeneous cosmological models in a variety of higher-dimensional theories containing Abelian and non-Abelian gauge fields. In particular, we study a five-dimensional theory containing an Abelian Maxwell field, a non-Abelian Yang-Mills model in six and N dimensions and a five-dimensional supergravity theory. We investigate a number of exact solutions of the corresponding field equations and study the asymptotic properties of more general solutions using qualitative techniques. The resulting models give rise to perfect fluid induced matter with a wide variety of equations of state. Finally, we investigate the introduction of (three-dimensional) spatial anisotropy into the higher-dimensional geometry, whereby the Einstein-Yang-Mills theories give rise to induced matter with non-perfect fluid contributions.

1. Introduction

Higher dimensions are believed to play a significant rôle in the early universe and there have been many recent attempts to construct a unified field theory based on the idea of a multidimensional spacetime [1–3]. There are several mechanisms known which incorporate a natural splitting of the physical and internal (higher) dimensions, including the Freund-Rubin mechanism [4], the Casimir effect associated with matter fields or zero-point gravitational energies [5], and the effect of higher-derivative terms in the gravitational action [6, 7]. Theories of this type date back to the original Kaluza-Klein theory [8–10] in which the extra degrees of freedom in a five-dimensional theory were associated with an electromagnetic potential and the resulting Einstein equations mimicked the Einstein-Maxwell equations in four dimensions. Modern theories of this type include supergravity theory [11, 12] and superstrings [13, 14].

In higher-dimensional theories (e.g., unifications of gravity with weak and strong interactions, as well as electromagnetism), when coordinates are chosen in such a way that the off-diagonal metric components are associated with gauge fields, an isometry group of internal compact manifolds generates a non-Abelian group of gauge transformations which lead to an effec-

[1]e-mail: hossein@mscs.dal.ca
[2]e-mail: jaf@mscs.dal.ca
[3]e-mail: aac@mscs.dal.ca

tive four-dimensional action for Einstein gravity plus non-Abelian gauge fields. Of course, finding solutions of the resulting field equations of these theories, in particular in the spatially homogeneous and isotropic case, is of interest in its own right. However, we shall also be interested in induced matter theories [15–17], in which the properties of matter are contained in a purely geometric Kaluza-Klein-type extension of general relativity and hence the matter is completely geometric in nature. Normally, the four-dimensional properties of matter are investigated by assuming that the higher-dimensional vacuum equations of general relativity reduce to Einstein's four-dimensional theory with matter [15, 16], although higher-dimensional generalized Lagrangian extensions of general relativity (with the addition of quadratic curvature invariants to the Einstein-Hilbert action) have also been studied [17]. Here we shall investigate whether Einstein's four-dimensional theory with matter can be embedded in a higher-dimensional theory of Yang-Mills-type, i.e., whether the correct field equations are the vacuum Einstein-Yang-Mills (EYM) equations. The idea is that the extra terms present in the higher-dimensional field equations may play the rôle of the matter terms that appear on the right-hand sides of the embedded four-dimensional Einstein field equations. The notion that the properties of matter might have a geometric origin has been developed by many authors [18–20] and is in the spirit of the original Kaluza-Klein theory [8–10].

We shall consider the $D=4+N$ dimensional metric in the form

$$ds^2 = g_{ab}dx^a dx^b = g_{\alpha\beta}dx^\alpha dx^\beta + g_{AB}dy^A dy^B, \quad (1)$$

where $ds_4^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ is given by the Friedmann-Robertson-Walker (FRW) form,

$$ds_{\rm F}^2 = -dt^2+H^2(t)\left[\frac{dr^2+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)}{\left(1+\frac{1}{4}kr^2\right)}\right], \quad (2)$$

where k is the normalized (i.e., $k=0,\pm1$) curvature constant. The matter source is assumed to be a perfect fluid with the energy-momentum tensor

$$T_{\alpha\beta} = (\mu+p)u_\alpha u_\beta + p g_{\alpha\beta}, \quad (3)$$

where μ and p are the energy density and pressure, respectively, and u^α is the (comoving) fluid four-velocity. The four-dimensional Einstein equations (with matter) then yield

$$8\pi G\mu = \frac{3}{H^2}(k+\dot{H}^2), \quad (4a)$$

$$8\pi G p = -2\frac{\ddot{H}}{H} - \frac{1}{H^2}(k+\dot{H}^2). \quad (4b)$$

We shall be primarily concerned with cosmological models containing a perfect fluid. The phenomenological physical quantities μ and p are to be interpreted in terms of more fundamental geometric quantities.

As an illustration, consider the five-dimensional metric given by

$$ds^2 = ds_{\rm F}^2 + L^2(t)dy^2, \quad (5)$$

where $ds_{\rm F}^2$ is the four-dimensional FRW line element given by (2) with $k=0$. The five-dimensional Einstein vacuum equations then yield [16]

$$8\pi G\mu = 3\frac{\dot{H}^2}{H^2} = -3\frac{\dot{H}}{H}\frac{\dot{L}}{L}, \quad (6)$$

$$8\pi G p = -2\frac{\ddot{H}}{H} - \frac{\dot{H}^2}{H^2} = \frac{\ddot{L}}{L} + 2\frac{\dot{H}}{H}\frac{\dot{L}}{L}, \quad (7)$$

with the familiar solution

$$H = t^{1/2}, \qquad L = t^{-1/2}, \quad (8)$$

$$\frac{8\pi G}{3}\mu = 8\pi G p = \frac{1}{4t^2}, \quad (9)$$

which represents the familiar flat FRW radiation model. In higher dimensions ($N>1$) with flat spatial curvature, the familiar generalized Kasner models are derived [17].

Here we wish to examine cosmological models in which Maxwellian and Yang-Mills terms are added to the standard Einstein Hilbert action

$$S = \int d^D V \frac{-R}{4\kappa^2}. \quad (10)$$

In particular, in Sec. 2.1 we augment (10) in five dimensions with an Abelian $U(1)$ field. Next we consider an $SO(3)$ model in D dimensions in Sec. 2.2. In Sec. 2.3 we derive the induced matter from a five-dimensional theory of supergravity. Sec. 3.1 gives an example of how one could generalize these examples by considering anisotropy in the three-space and Sec. 4 is reserved for concluding remarks.

2. Einstein-Yang-Mills theories

2.1. Abelian gauge field

We begin by examining five-dimensional Einstein gravity augmented by a Maxwellian field, described by the action

$$S = \int d^5 V\left\{-\frac{(R+2\Lambda)}{4\kappa^2} - \frac{1}{4}F_{ab}F^{ab}\right\}, \quad (11)$$

where $\kappa^2 = 4\pi^{(5)}G$ and F_{ab} is the field tensor of a $U(1)$ Abelian guage field. This model has been extensively studied in [21] and [22], in which the Rubin-Freund ansatz

$$F = \frac{QL}{4\pi H^3} dt\wedge dy, \quad (12)$$

has been assumed and where the metric is given by (5).

By varying the action (11) we obtain the following relevant field equations:

$$\frac{\ddot{L}}{L} + 3\frac{\ddot{H}}{H} = \frac{2\Lambda}{3} - \frac{{}^{(5)}GQ^2}{3\pi H^6}, \tag{13a}$$

$$\frac{\ddot{L}}{L} + 3\frac{\dot{H}\dot{L}}{HL} = \frac{2\Lambda}{3} - \frac{{}^{(5)}GQ^2}{3\pi H^6}, \tag{13b}$$

$$\frac{\ddot{H}}{H} + \frac{\dot{H}\dot{L}}{HL} + \frac{2(\dot{H}^2+k)}{H^2} = \frac{2\Lambda}{3} + \frac{{}^{(5)}GQ^2}{6\pi H^6}, \tag{13c}$$

where a dot denotes d/dt. In [21] these field equations were used to describe an N-dimensional compact internal space with two additional dimensions: one timelike and one spacelike. This particular example then implies that $k > 1$ in the above field equations. However, since the internal space is spacelike, one could equally view this as describing a space-time with one timelike dimension, three spacelike dimensions (by setting $n = 3$) and one internal dimension, and therefore k can be 0 or ± 1.

A solution to (13) for H and L is given by

$$\dot{H}^2 = L^2 = \frac{\sqrt{2\,{}^{(5)}GM}}{H^2}\sqrt{H^2 - \frac{Q^2}{24\pi M}} + \frac{2\Lambda H^6}{24\,{}^{(5)}GM} - \frac{kH^4}{2\,{}^{(5)}GM} \tag{14}$$

where M is an integration constant. When $\Lambda = k = 0$, and defining α and β by $\alpha = \sqrt{2\,{}^{(5)}GM}$, $\beta^2 = Q^2[24\pi M]^{-1}$, Eq. (14) can be further integrated to yield

$$\alpha t = \tfrac{1}{2}\beta^2 \cosh^{-1}(H/\beta) + \tfrac{1}{2}H\sqrt{H^2-\beta^2}, \tag{15}$$

although it is not necessary to determine H in terms of t in order to determine the forms for μ and p, which are given by

$$8\pi G\mu = \frac{6\,{}^{(5)}GM}{H^4} - \frac{{}^{(5)}GQ^2}{4\pi H^6}, \tag{16}$$

$$8\pi Gp = \frac{2\,{}^{(5)}GM}{H^4} - \frac{{}^{(5)}GQ^2}{4\pi H^6}. \tag{17}$$

Hence, we obtain the equation of state

$$(\mu - 3p)^2 = \omega^2(\mu - p)^3 \tag{18}$$

where

$$\omega^2 \equiv \frac{GQ^4}{32\pi\,{}^{(5)}GM^3}.$$

The late-time equation of state for these solutions is that of radiation. This is apparent from (15), since for large H we obtain $H^2 \approx 2\alpha t$, which leads to $8\pi Gp \approx \frac{83}{\pi}G\mu \approx \frac{1}{4}t^{-2}$. The corresponding line element is then

$$ds^2 \approx -dt^2 + 2\alpha t(dr^2 + r^2 d\Omega^2) + \frac{\alpha}{2t}dy^2,$$

whose four-dimensional component is the Tolman line element [23].

2.2. Yang-Mills fields in higher dimensions

We now turn our attention to *non-Abelian* fields coupled to higher-dimensional Einstein gravity. As an explicit example, we shall consider an $SO(3)$ Yang-Mills field coupled to gravity via the six-dimensional action [14]

$$S = \int d^6V \left\{ -\frac{(R+2\Lambda)}{4\kappa^2} - F^{(a)}_{\alpha\beta}F^{(a)\alpha\beta} \right\} \tag{19}$$

(where now $\kappa^2 = 4\pi\,{}^{(6)}G$), with the metric described by the line interval

$$ds^2 = ds_F^2 + L^2(t)\left[d\xi^2 + \sin^2(\xi)d\zeta^2\right]$$

where $y^1 = \xi$ and $y^2 = \zeta$ are the two extra coordinates. We assume also that all components of the gauge field are zero except

$$\begin{aligned} A^{(a)}_\xi &= -e^{-1}\left[-\sin\zeta, \cos\zeta, 0\right], \\ A^{(a)}_\zeta &= -e^{-1}\sin\xi\left[-\cos\zeta\cos\xi, \sin\zeta\cos\xi, \sin\xi\right], \end{aligned} \tag{20}$$

where e is the gauge field strength (see [14]).

The relevant field equations obtained from this action using the above ansätze for the metric and the gauge field, are

$$3\frac{\ddot{H}}{H} + 2\frac{\ddot{L}}{L} = \frac{1}{2}\Lambda - \frac{\alpha}{L^4}, \tag{21a}$$

$$\frac{\ddot{H}}{H} + 2\frac{\dot{H}\dot{L}}{HL} + 2\frac{\dot{H}^2}{H^2} + 2\frac{k}{H^2} = \frac{1}{2}\Lambda - \frac{\alpha}{L^4}, \tag{21b}$$

$$\frac{\ddot{L}}{L} + 3\frac{\dot{H}\dot{L}}{HL} + \frac{L^2}{\dot{L}^2} + \frac{K}{L^2} = \frac{1}{2}\Lambda + \frac{3\alpha}{L^4}, \tag{21c}$$

$$\frac{3\dot{H}^2}{H^2} + \frac{3k}{H^2} + 6\frac{\dot{H}\dot{L}}{HL} + \frac{\dot{L}^2}{L^2} + \frac{K}{L^2} = \Lambda + \frac{2\alpha}{L^4} \tag{21d}$$

where $\alpha = 2\pi\,{}^{(6)}G/e^2$. Cremmer and Scherk [14, 24] presented a six-dimensional Yang-Mills solution similar to the 't Hooft magnetic monopole [25, 26] where $L = L_0$ a constant and $k = 0$ (see also [27]). Their solution is a fixed point of the system (21) for $N = 2$, $K = 1$ and $k = 0$ (see below).

These field equations can be generalized to the $(4+N)$-dimensional case in a straightforward manner. In particular, we can exploit the results of Wiltshire [22] who studied an Abelian gauge field (using the Rubin-Freund ansatz) coupled to $(4+N)$-dimensional gravity using the line interval

$$ds^2 = ds_{\rm F}^2 + L^2(t)\tilde{g}_{IJ}dy^I dy^J.$$

Here the "internal" space is an N-dimensional Einstein space of constant curvature K, described by the metric $\tilde{g}_{IJ}$; i.e., the Ricci tensor constructed from $\tilde{g}_{IJ}$ is defined by $R_{IJ} = (N-1)K\tilde{g}_{IJ}$.

The field equations in [22] are given by

$$3\frac{\ddot{H}}{H}+N\frac{\ddot{L}}{L}=\frac{2\Lambda}{N+2}-\frac{\alpha(N-1)}{L^{2N}}, \tag{22a}$$

$$\frac{\ddot{H}}{H}+N\frac{\dot{H}\dot{L}}{HL}+\frac{2}{H^2}\left(\dot{H}^2+k\right)=\frac{2\Lambda}{N+2}-\frac{\alpha(N-1)}{L^{2N}}, \tag{22b}$$

$$\frac{\ddot{L}}{L}+3\frac{\dot{H}\dot{L}}{HL}+\frac{N-1}{L^2}\left(\dot{L}^2+K\right)=\frac{2\Lambda}{N+2}+\frac{3\alpha}{L^{2N}}, \tag{22c}$$

$$6\left(\frac{\dot{H}^2+k}{H^2}\right)+6N\frac{\dot{H}\dot{L}}{HL}+N(N-1)\left(\frac{\dot{L}^2+K}{L^2}\right)=2\Lambda+\frac{(N+2)\alpha}{L^{2N}}. \tag{22d}$$

Our immediate focus here is with the case of an $SO(3)$ non-Abelian guage field in six dimensions. However, we note that the results pertaining to this particular case follow immediately from the solutions of (22) by setting $N=2$ (for more details about the $SO(3)$ model see [28]). Attempts to solve (22) analytically for the most general solution may prove futile. However, the behaviour of the system for all times may be obtained through qualitative analysis, as was the method used by Wiltshire. In his work, he completed a full phase-space analysis of Eqs. (22), including the use of a Poincaré transformation to compactify the space in order to evaluated the system's fixed points at infinity (in terms of the dynamical variables used). We only highlight the solutions obtained in Wiltshire's work and refer the reader to his paper for full details of the analysis. Specifically, we will describe the non-saddle fixed points of (22), present the "induced" equation of state associated with each of these fixed points, and then briefly summarize the behaviour of the solutions.

The field equations (22) admit up to seven non-saddle fixed points, although some of these fixed points are described by identical solutions. The fixed points forming the first set are the only ones at infinity and are represented by the generalized Kasner solution [17, 22, 29, 30]

$$H=H_0\,t^{m_\pm}, \qquad L=L_0\,t^{n_\pm}, \tag{23}$$

where

$$m_\pm = \frac{1}{3+N}\left\{1\pm\frac{1}{3}\sqrt{3N^2+6N}\right\} \gtrless 0, \tag{24}$$

$$n_\pm = \frac{1}{3+N}\left\{1\mp\frac{1}{N}\sqrt{3N^2+6N}\right\} \lessgtr 0, \tag{25}$$

where two of the fixed points have (m_+,n_+) as a solution and the other two have (m_-,n_-) as a solution. The latter two points are saddle points and so solutions asymptoting towards or away from these are of measure zero. One of the m_+ and n_+ fixed points is an attracting node whilst the other is a repelling node. Here, and throughout the rest of the paper, when the Kasner solution is mentioned, it will be assumed that we are referring to the m_+ and n_+ solution unless otherwise stated.

For both solutions $(m_\pm,n_\pm)$, the induced matter has the equation of state (see [17])

$$p=\sigma_\pm\mu=\left\{\frac{2N+3\mp\sqrt{3N^2+6N}}{3\pm\sqrt{3N^2+6N}}\right\}\mu. \tag{26}$$

For the (m_+,n_+) solution, σ_+ ranges from $\frac{1}{3}$ for $N=1$ to $\frac{2}{3}(\sqrt{3}-1)$ for $N\to\infty$.

The next two fixed points are obtained for $k=K=\alpha=0$ and $\Lambda>0$, and are represented by the solutions

$$H=H_0\,e^{\gamma t}, \qquad L=L_0\,e^{\gamma t}, \tag{27}$$

where

$$\gamma=\pm\sqrt{\frac{2\Lambda}{(N+2)(N+3)}}. \tag{28}$$

The growing mode solution is an attracting node and hence a future attractor, whilst the decaying mode solution is a repelling node (past attractor). The induced energy density and pressure for this solution are given by

$$8\pi G\mu=-8\pi Gp=3\frac{2\Lambda}{(N+2)(N+3)}. \tag{29}$$

This equation of state corresponds to that of a false vacuum and so we have de Sitter-like solutions. The next set of fixed points is another set of de Sitter-like solutions for $k=0$, $K=1$, $\Lambda>0$. Although the number of fixed points is either two or four depending on the value of Λ, they all correspond to the solution

$$H=H_0\,e^{\pm\delta t}, \qquad L=L_0. \tag{30}$$

This is the form of the solution obtained by Cremmer and Scherk [14]. The integration constant L_0 is not arbitrary, but depends on the values of Λ and α. In all these cases, the equation of state is again $p=-\mu$. If $\alpha=0$, then there are only two solutions: $\delta^2=\Lambda/6$ with $L_0^{-2}=\frac{1}{2}\Lambda$. When $\alpha\neq 0$, finding δ and L_0 in a closed form may be quite difficult for arbitrary N. To illustrate, for these solutions the first two equations of (22) both yield $(N\neq 1)$

$$\frac{1}{L_0^{2N}}=\frac{2\Lambda-4(N+2)\delta^2}{\alpha(N-1)(N+2)},$$

which isolates the value for L_0. Using this expression, one then obtains from either of the last two equations of (22)

$$\alpha\left[2\Lambda-9\delta^2\right]^N=\frac{(N-1)^{2N-1}}{(N+2)}\left[2\Lambda-3(N+2)\delta^2\right],$$

which is the condition found in [22]. Unfortunately, one cannot analytically solve this for arbitrary N. However, to demonstrate that this does lead to either two or four solutions, we shall consider the case $N = 2$. We find that Λ is bounded by $\Lambda \leq (6\alpha)^{-1}$ for any real solution to exist, so we write $\Lambda = z/(6\alpha)$ where z has the range $[0, 1]$. The solution for δ and L_0 is hence

$$\delta_\pm^2 = \frac{2z - 1 \pm \sqrt{1-z}}{54\alpha}, \tag{31}$$

$$\frac{1}{L_0^4} = \frac{1 - \frac{1}{2}z \mp \sqrt{1-z}}{18\alpha^2}. \tag{32}$$

It is apparent that there is no real solution for $z > 1$ ($\Lambda > (6\alpha)^{-1}$). For $\Lambda \geq 1/(8\alpha)$ we find that the δ_+ solution is an attracting node for $\delta > 0$ and a repelling node for $\delta < 0$, and the δ_- solutions are saddle points of the system. For $\Lambda < 1/(8\alpha)$, the δ_-^2 solutions are not real and so there are only two fixed points (δ_+^2) which are saddle points.

The final fixed point is given by the solution $k = K = \alpha = \Lambda = \dot{L} = \dot{H} = 0$, which is just a D-dimensional Minkowski space-time. This fixed point is an attracting node.

From the dynamical systems analysis [22], or from a direct perturbation analysis [28], we find the following evolution of this system. With the exception of solutions of measure zero, all solutions asymptote into the past to the Kasner solution, or to the decaying de Sitter solution of (27), or to the decaying de Sitter solution for δ_- of (31) (for the correct values of Λ). The future behaviour of the solutions is that they either recollapse to the Kasner singularity (i.e.,the time reverse solution of (23) with m_+ and n_+), or they asymptote towards the growing de Sitter solution (27), or to the growing de Sitter solution of (31) for δ_-, or to Minkowski spacetime.

Finally, Wiltshire [22] finds three exact solutions which represent separatrices in the phase portraits constructed in his analysis. The first is the Kasner solution (23) when $k = K = \alpha = \Lambda = 0$. The induced matter is characterized by (26). The next two solutions occur for $k = K = \alpha = 0$. The scale factors H and L for the first of these solution, which were obtained for $\Lambda < 0$, are

$$H = H_0 \left|\sin(\tfrac{1}{2}\gamma t)\right|^{m_\pm} \left|\cos(\tfrac{1}{2}\gamma t)\right|^{\frac{2}{N+3} - m_\pm}, \tag{33}$$

$$L = L_0 \left|\sin(\tfrac{1}{2}\gamma t)\right|^{n_\pm} \left|\cos(\tfrac{1}{2}\gamma t)\right|^{\frac{2}{N+3} - n_\pm} \tag{34}$$

where

$$\gamma^2 = 2\left[\frac{N+3}{N+2}\right]|\Lambda|.$$

The corresponding energy density and pressure are, respectively,

$$8\pi G\mu = \frac{|\Lambda|}{2(N+2)}\Big\{(j_\mu \pm l)\cot^2(\tfrac{1}{2}\gamma t) + (j_\mu \mp l)\tan^2(\tfrac{1}{2}\gamma t) + 2(N-1)\Big\}, \tag{35}$$

$$8\pi G p = \frac{|\Lambda|}{6(N+2)}\Big\{(j_p \pm l)\cot^2(\tfrac{1}{2}\gamma t) + (j_p \mp l)\tan^2(\tfrac{1}{2}\gamma t) - 6(N-3)\Big\}, \tag{36}$$

where

$$l = \frac{2\sqrt{3N^2+6N}}{N+3}, \quad j_\mu = \frac{N^2+2N+3}{N+3}, \quad j_p = \frac{9-3N^2}{N+3}.$$

These solutions initially expand from a Kasner singularity and then recollapse back to a Kasner singularity (see [22]) and so both early time ($t \to 0^+$) and late time ($\gamma t \to \pi$) solutions will have the Kasner equation of state (26).

The last of the separatrix solutions has $\Lambda > 0$, and the scale factors are given by

$$H = H_0 \left|\sinh(\tfrac{1}{2}\gamma t)\right|^{m_\pm} \left|\cosh(\tfrac{1}{2}\gamma t)\right|^{\frac{2}{N+3} - m_\pm}, \tag{37}$$

$$L = L_0 \left|\sinh(\tfrac{1}{2}\gamma t)\right|^{n_\pm} \left|\cosh(\tfrac{1}{2}\gamma t)\right|^{\frac{2}{N+3} - n_\pm}. \tag{38}$$

The corresponding energy density and pressure are, respectively,

$$8\pi G\mu = \frac{|\Lambda|}{2(N+2)}\Big\{(j_\mu \pm l)\coth^2(\tfrac{1}{2}\gamma t) + (j_\mu \mp l)\tanh^2(\tfrac{1}{2}\gamma t) - 2(N-1)\Big\}, \tag{39}$$

$$8\pi G p = \frac{|\Lambda|}{6(N+2)}\Big\{(j_p \pm l)\coth^2(\tfrac{1}{2}\gamma t) + (j_p \mp l)\tanh^2(\tfrac{1}{2}\gamma t) + 6(N-3)\Big\}. \tag{40}$$

For early times, $t \to 0^+$, the equation of state approaches the Kasner equation of state, whereas at late times, $\gamma t >> 1$, the equation of state approaches the false vacuum equation (29).

2.3. Supergravity

In this section, we study an example from supergravity in which the fermionic fields are zero, the Maxwellian potential is given by $A_\lambda = (0, 0, 0, 0, \psi)$ and the five-dimensional line interval is given by

$$ds^2 = -dt^2 + H^2(t)\frac{dr^2 + r^2 d\Omega^2}{[1 + \frac{1}{4}kr^2]^2} + L^2(t)dy^2 = -H^6(\eta)d\eta^2 + H^2(\eta)d\Sigma_3^2 + L^2(\eta)dy^2 \tag{41}$$

where the conformal time coordinate is defined by $dt \equiv H^3 d\eta$. As given in [31,32], all quantities considered here depend only on the four-dimensional "external" coordinates and so the five-dimensional Lagrangian can be expressed as a four-dimensional Lagrangian coupled to two scalar fields, ψ and $L^2 = g_{yy}$, namely,

$$S = \int d^4V \left\{ -\frac{LR}{4\kappa^2} + \frac{2}{L} D_\lambda \psi D^\lambda \psi \right\} \tag{42}$$

where D_λ is the gauge-covariant derivative corresponding to A_λ.

The resulting field equations,

$$\frac{\ddot{H}}{H} + \frac{\dot{H}^2}{H^2} + \frac{k}{H^2} = \frac{\dot{\psi}^2}{4L^2}, \tag{43a}$$

$$\frac{\dot{H}^2}{H^2} + \frac{k}{H^2} + \frac{\dot{H}\dot{L}}{HL} = \frac{\dot{\psi}^2}{4L^2}, \tag{43b}$$

$$\ddot{L} + 3\frac{\dot{H}}{H}\dot{L} = -\frac{\dot{\psi}^2}{L^2}, \tag{43c}$$

$$\ddot{\psi} + 3\frac{\dot{H}}{H}\dot{\psi} = \frac{\dot{L}}{L}\dot{\psi}, \tag{43d}$$

were solved in [31, 32]; the solution, up to a translation in η, is given by

$$H = \frac{H_0}{\sqrt{1 - q\cos(a\eta)}}, \tag{44}$$

$$L = -L_0 \sin(a\eta), \tag{45}$$

$$\psi = -L_0 \cos(a\eta), \tag{46}$$

where H_0, L_0 and a are integration constants and

$$q \equiv \sqrt{1 - 4kH_0^4/a^2}. \tag{47}$$

To ensure that t is monotonic in η, H is required to be positive, as is the case when $0 < q < 1$. It is apparent that H oscillates all the time (and therefore μ and p will oscillate all the time). When $q \geq 1$, t and H diverge for $\eta = 1/a \cos^{-1}(1/q)$, as pointed out in [31, 32]. Except for the trivial case $H_0 = 0$, H never vanishes and so the four-dimensional space-time can be considered to be singularity-free.

The energy density and pressure of the induced matter are given by, respectively,

$$8\pi G\mu = \frac{3}{4}\frac{a^2}{H_0^6}\{1 - q\cos(a\eta)\} \times \left\{(1-q) + q^2\sin^2(a\eta)\right\}, \tag{48}$$

$$8\pi Gp = \frac{a^2}{H_0^6}\{1 - q\cos(a\eta)\} \times \left\{ q\cos(a\eta)[1 - q\cos(a\eta)] - \frac{1}{4}q^2\sin^2(a\eta) - \frac{1}{4}(1-q) \right\}. \tag{49}$$

We define $\bar{p} = 8\pi G p H_0^6 a^{-2}$ and $\bar{\mu} = 8\pi G \mu H_0^6 a^{-2}$, and combine (48) and (49) to obtain the equation of state

$$\left\{ \frac{27(\bar{\mu} - \bar{p})^5}{(\bar{\mu} + 3\bar{p}) - 3C(\bar{\mu} - \bar{p})} + C(\bar{\mu} + 3\bar{p}) - \frac{12\bar{\mu}(\bar{\mu} - \bar{p})^2}{(\bar{\mu} + 3\bar{p}) - 3C(\bar{\mu} - \bar{p})} \right\}(\bar{\mu} - \bar{p}) = 0, \tag{50}$$

where $C = 1 - q + q^2 > 0$.

To help elucidate the nature of this equation of state, we have provided several figures of p, μ and p/μ as function of η for various values of q. In the calculations used to produce the figures, we defined $t = 0$ for $\eta = \pi/a$. In the plots for $q < 1$, the value of μ and p repeat themselves every $2\pi/a$ and so we only plot them from $\eta = 0$ to $\eta = 2\pi/a$. For $q \geq 1$, we only plot μ and p for the range of η which corresponds to $t \in (-\infty, \infty)$ (which are marked on the plots by the dashed lines). For these values of q, the equation of state asymptotes into the past *and* future towards the relation $p = -\frac{1}{3}\mu$.

Fig. 1 gives the plots of the energy density and pressure for the induced matter derived in the supergravity model for the following values of the parameter q (defined by Eq. (47)): (a) $q < 1/4$, (b) $q = 1/4$, (c) $q = 1/2$, (d) $q = 3/4$, (e) $q > 1$.

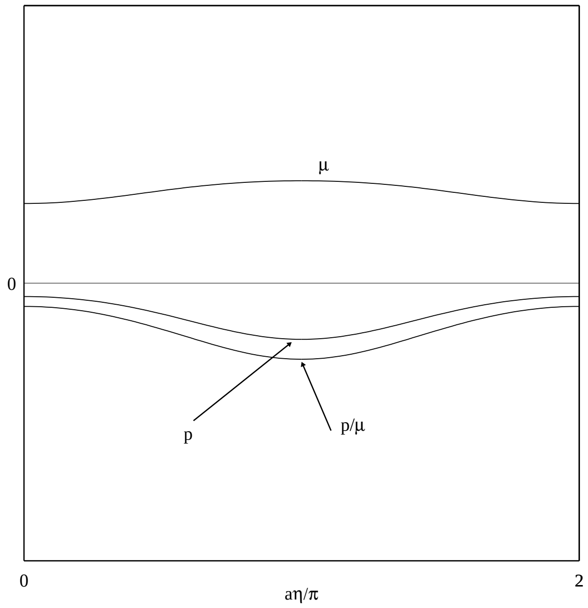

Figure 1(a): $q < \frac{1}{4}$

3. Generalizations

From these examples it is quite apparent that there are many different ways of obtaining equations of state different from radiation in the context of induced matter theory. Indeed, there are more examples and theories found in the literature that may be used in this manner. In the context of Einstein-Maxwell (EM) theories, Gleiser *et al.* [33] have studied ten- and eleven-dimensional space-times, and Freund and Rubin [4] have also found solutions in the eleven-dimensional case in which seven of the eleven dimensions compactify. Gibbons and Wiltshire [21] studied arbitrary D-dimensional spacetimes containing an EM gauge field.

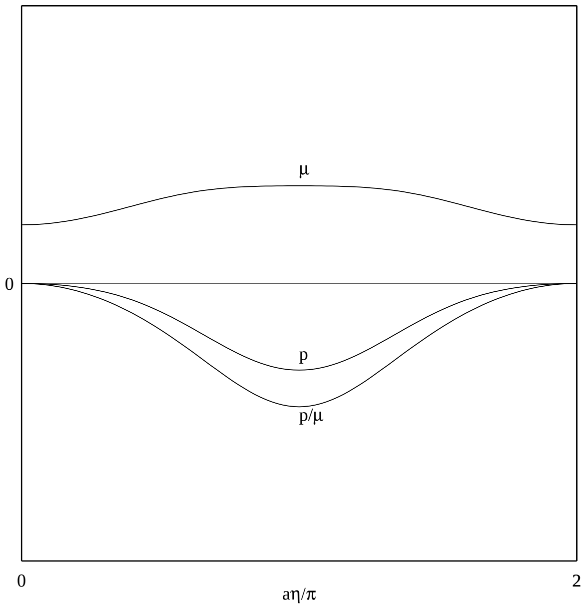

Figure 1(b): $q = \frac{1}{4}$

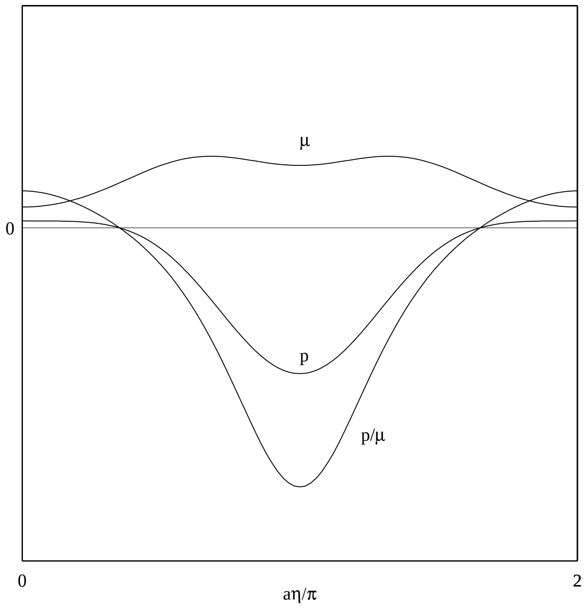

Figure 1(c): $q = \frac{1}{2}$

Similarly, Fabris [34] showed that in order to obtain a traceless electromagnetic stress-energy tensor in $D = 4 + N$ dimensions, the electromagnetic potential is required to have a $\frac{1}{2}(N-2)$-form, and hence he considered even-dimensional cosmologies. Fabris [35] also studied a $D = 6$ anisotropic model and a $D = 8$ model which contained anti-de Sitter space-time as a solution.

In terms of Einstein-Yang-Mills higher-dimensional theories, the literature is extensive. Kubyshin *et al.* [36] studied higher-dimensional cosmologies containing $SU(5)$ and $SU(2) \times U(1)$ gauge fields with a static compact "internal" space, as well as anisotropic internal spaces. Clément [37] studied a six-dimensional $SO(3)$ EYM-Higgs model, examining the stability of the static solutions. Bertolami *et al.* [38] considered D-dimensional space-times in the context of compactification. Luciani [39] extended the work of Cremmer and Scherk [14, 24] by considering several symmetry groups [for example, a $(4 + 2PQ)$-dimensional space-time with the group $SU(P+Q)$ and the subgroup $SU(P) \times SU(Q) \times U(1)$, a $(4 + \frac{1}{2}(N-1)(N+2))$-dimensional spacetime with the group $SU(N)$ and the subgroup $SO(N)$, a $(4+N(N-1))$-dimensional space-time with the group $SO(2N)$ and the subgroup $U(N)$, and a $(4+PQ)$-dimensional space-time with the group $SO(P+Q)$ and the subgroup $SO(P) \times SO(Q)$].

There are several examples of supergravity theories that have been studied. Five-dimensional supergravity has been studied by Balbinot *et al.* [31, 32] and by Pimentel [40] (who considered a Bianchi I model for the four-dimensional part of the space-time). In addition, Duruisseau and Fabris [41] studied five-dimensional supergravity with Gauss-Bonnet terms in the action. Ten-dimensional supergravity has also been studied by Gleiser and Stein-Schabes [42], who obtained a de Sitter-type solution as a late time solution.

Of course, there are other approaches one could consider. For instance, one could also propose a generalized Einstein theory of gravity in the context of Lovelock theory [17, 43]. Another example could be to include anisotropy into any of the aforementioned works. In all examples studied in Sec. 2 the induced fluids were perfect; by introducing anisotropy into the three-space we would expect inducing anisotropies in the pressure and hence dissipative terms in the energy-momentum tensor. In general, the energy-momentum tensor would be then modified from (3) to

$$T_{\alpha\beta} = (\mu + p)u_\alpha u_\beta + p g_{\alpha\beta} + \pi_{\alpha\beta} + q_\alpha u_\beta + q_\beta u_\alpha \tag{51}$$

where $\pi_{\alpha\beta}$ is the anisotropic pressure tensor and q^α is the heat conduction vector, such that $\pi^\alpha_\alpha = u_\alpha \pi^\alpha_\beta = q^\alpha u_\alpha = 0$ [44]. The variable p is now the pressure averaged over all three directions and the pressure in each direction is then defined as $p_i = p + \pi^i_i$ (for $i = 1, 2, 3$ with no summation implied).

3.1. Anisotropic generalizations

As an illustration, we consider anisotropy in the supergravity model of Sec. 2.3 which has been previously studied in [31]. The cylindrically symmetric metric is given by

$$\begin{aligned} ds^2 = &-A^2 B^4 d\eta^2 + A^2 dx^2 \\ &B^2(dy^2 + dz^2) + L^2(dx^5)^2 \end{aligned} \tag{52}$$

where now the conformal time, η, is defined by $dt = AB^2 d\eta$, with $A = A(\eta)$, $B = B(\eta)$ and $L = L(\eta)$.

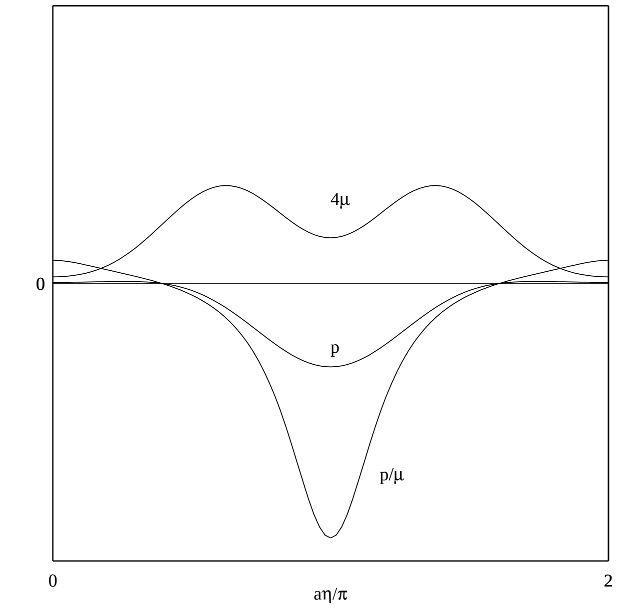

Figure 1(d): $q = \frac{3}{4}$

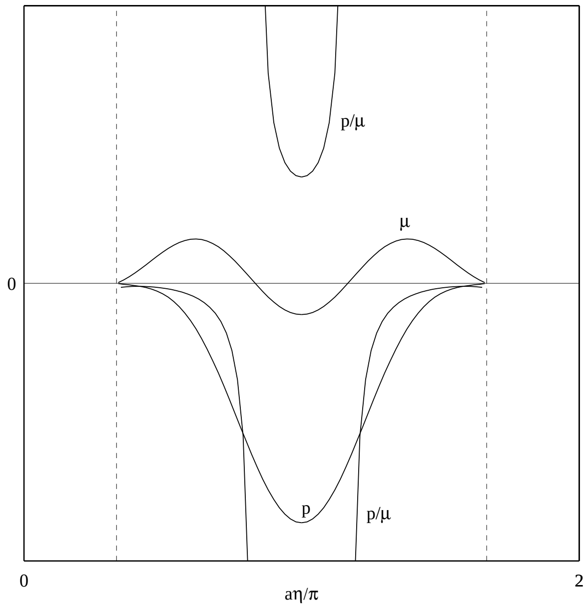

Figure 1(e): $q > 1$

The field equations then give rise to the following set of ordinary differential equations (see [31] for details):

$$\frac{1}{2}\frac{\psi'}{L^2} = \frac{2A'}{A}\frac{B'}{B} + \frac{(B')^2}{B^2} + \left[\frac{A'}{A} + \frac{2B'}{B}\right]\frac{L'}{L}, \qquad (53a)$$

$$\frac{1}{2}\frac{\psi'}{L^2} = -\frac{2B''}{B} - \frac{L''}{L} + \frac{2B'}{B}\frac{A'}{A} + \frac{3\,(B')^2}{B^2} + \frac{L'}{L}\frac{A'}{A}, \qquad (53b)$$

$$\frac{1}{2}\frac{\psi'}{L^2} = -\frac{A''}{A} - \frac{B''}{B} - \frac{L''}{L} + \frac{(A')^2}{A^2} + \frac{2\,(B')^2}{B^2} + \frac{L'}{L}\frac{B'}{B}, \qquad (53c)$$

$$\frac{1}{2}\frac{\psi'}{L^2}\frac{A''}{A} + \frac{2B''}{B} - \frac{2A'}{A}\frac{B'}{B} - \frac{3\,(B')^2}{B^2} - \frac{(A')^2}{A^2}, \qquad (53d)$$

$$\psi'' - \frac{L'}{L}\psi' = 0. \qquad (53e)$$

A solution of these equations, up to a translation in η, is then given by [31]

$$\begin{aligned} A &= A_0\left(\tan\frac{a\eta}{2}\right)^{b/a}\frac{1}{\sqrt{\sin(a\eta)}}, \\ B &= B_0\left(\tan\frac{a\eta}{2}\right)^{c/a}\frac{1}{\sqrt{\sin(a\eta)}}, \\ L &= L_0\sin(a\eta), \\ \psi &= -L_0\cos(a\eta), \end{aligned} \qquad (54)$$

where the integration constants a, b and c are constrained by $2bc + c^2 = 3a^2/4$. Evidently, when $b = c = \pm a/2$, the solutions found in Sec. 2.3 are recovered.

To calculate the induced matter, we define the comoving fluid four-velocity to be

$$u^\alpha = \frac{\delta^\alpha_t}{A(\eta)B^2(\eta)},$$

which satisfies $u^\alpha u_\alpha = -1$. From (51) we then obtain

$$\begin{aligned} 8\pi G\mu &= \frac{1}{4A_0^2B_0^4}\left[\tan\left(\tfrac{1}{2}a\eta\right)\right]^{-2\frac{b+2c}{a}}\sin(a\eta) \\ &\quad\times[2c - a\cos(a\eta)]\sin(a\eta) \\ &\quad\times[4b + 2c - 3a\cos(a\eta)], \end{aligned} \qquad (55)$$

$$\begin{aligned} 8\pi Gp &= \frac{a}{12A_0^2B_0^4}\left[\tan\left(\tfrac{1}{2}a\eta\right)\right]^{-2\frac{b+2c}{a}} \\ &\times\left[9a\cos^2(a\eta) - 4(b+2c)\cos(a\eta) - 3a\right]\sin(a\eta), \end{aligned} \qquad (56)$$

$$\begin{aligned} 8\pi G\pi^x_x &= \frac{-2a}{3A_0^2B_0^4}\left[\tan\left(\tfrac{1}{2}a\eta\right)\right]^{-2\frac{b+2c}{a}}[b - c] \\ &\quad\times\sin(a\eta)\cos(a\eta), \end{aligned} \qquad (57)$$

$$\pi^y_y = \pi^z_z = -\tfrac{1}{2}\pi^x_x. \qquad (58)$$

Notice that there are no heat conduction terms in this model. It may be verified that $\pi^\alpha_\beta = -\lambda(\eta)\sigma^\alpha_\beta$, where σ^α_β is the shear tensor defined from u^α [44] and λ is the viscosity coefficient of the fluid given by

$$\lambda(\eta) = -\frac{a\cos(a\eta)\sqrt{\sin(a\eta)}}{A_0B_0^2}\left[\tan\left(\tfrac{1}{2}a\eta\right)\right]^{-\frac{b+2c}{a}}. \qquad (59)$$

We note that the above quantities either diverge or vanish at $\eta = 0$ and $\eta = \pi/a$, but which possibility occurs depends on the values of b and c, which can be positive or negative. One finds by iteration that the original time variable, defined by $t = \int AB^2 d\eta$, is monotonic in η in the interval $[0, \pi/a]$ and so we will consider these "endpoints" as early-time and late-time limits. Taking the ratio of p/μ and using the constraint $2bc + c^2 = 3a^2/4$, we obtain

$$\frac{p}{\mu} = \frac{a^2 cos(a\eta) + 2ac[1 - 3\cos^2(a\eta)] + 4c^2\cos(a\eta)}{3a[a - 2c\cos(a\eta)][a\cos(a\eta) - 2c]}.$$

For early-time ($\eta \to 0^+$) behaviour *and* for late-time behaviour ($\eta \to \pi/a$) we find that $p \to \mu/(3a)$ and hence the constant a plays an important rôle in determining the equation of state.

4. Conclusion

In this paper our main goal has been to consider the induced matter theory of Wesson [15] in the context of higher-dimensional Einstein-Yang-Mills cosmological models. We have studied Abelian and non-Abelian gauge fields coupled with gravity in $4+1$ and $4+N$ dimensions, respectively. These gauge fields do not arise from the metric (e.g., $A_\mu \propto g_{5\mu}$) as happens in the traditional Kaluza-Klein theory. In the case of an Abelian Maxwell field the induced matter is a perfect fluid with the equation of state (18). At late times the equation of state for this form of matter asymptotes towards that of radiation.

In the case of the non-Abelian Yang-Mills model, we described the fixed-point solutions of the field equations consisting of an autonomous system of ordinary differential equations, and we discussed the induced equation of state associated with these fixed-point solutions. We also gave the form of the solutions explicitly for two of the fixed points whose existence was simply noted in Wiltshire's work [22]. The general behaviour of the solutions is that they evolved either from anti-de Sitter spacetime or from a Kasner singularity. The solutions asymptote either towards another Kasner singularity, or towards a de Sitter inflationary phase, or towards a flat D-dimensional Minkowski vacuum at late times. The induced equation of state for these fixed points is linear but depends on the number of dimensions considered.

We also investigated the induced matter theory in the context of a 5-dimensional supergravity theory. The induced matter obtained is somewhat exotic but still of a perfect fluid form. For suitable values of the parameters we found that there are no initial singularities in the four-dimensional spacetime, which exhibits a periodic nature. Consequently, there are no early/late time behaviours for the induced matter. Instead, the energy density and the pressure have an oscillatory behaviour, the latter remaining mostly negative. For other values of the model parameters, there are indeed singularities and the matter asymptotes to $p = -\frac{1}{3}\mu$ for early and late times. In these cases there are times at which the energy density actually becomes negative.

Finally, by introducing anisotropy into the four-dimensional part of the spacetime, dissipation terms are added to the induced matter. Consequently, anisotropic pressures are introduced which are proportional to the fluid's shear. The corresponding viscosity coefficient either diverges or vanishes at early/late times, depending on the values of the parameters b and c. The induced matter has the asymptotic form $p = \mu/(3a)$ for early and late times.

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