Cylindrically symmetric space-times in 5D and their effective properties of matter (*)

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Summary. — Following recent work by Senovilla *et al.* on cylindrically symmetric 4D space-times, we give several exact solutions of this type in 5D and derive their 4D properties.

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1. – Introduction

Recently, considerable interest was generated by the discovery by Senovilla and others of exact cylindrically symmetric cosmological solutions of the field equations in 4D that are free of singularities [1-4]. The generalization of these solutions from 4D to 5D has been studied by Banerjee *et al.* [5]. It is now known that any solution of the apparently empty field equations in 5D can be interpreted as a solution of the 4D equations with an effective energy-momentum tensor induced by the extra dimension [6]. Thus from any exact solution in 5D one can obtain the properties of matter in 4D. This procedure has been applied to a number of cosmological metrics [7-12], some of which have singularities and some of which do not [13-15]. In this paper we will present several cylindrically symmetric solutions of the 5D field equations, evaluate their singularities, and derive their 4D properties of matter.

2. – Metrics and matter

We consider a metric of the type used by Banerjee *et al.* [5], namely

$$\mathrm{d}s^2 = A^2(\mathrm{d}t^2 - \mathrm{d}r^2) - B^2\,\mathrm{d}y^2 - C^2\,\mathrm{d}z^2 - D^2\,\mathrm{d}\psi^2. \tag{1}$$

Here the metric coefficients A, B, C, D can depend on the radius r and the time t, but not on the other spatial coordinates y, z and the Kaluza-Klein coordinate ψ. We use units throughout such that the speed of light c and the gravitational constant G obey $c = 1$, $8\pi G = 1$. We will consider solutions of the 5D field equations in apparent vacuum, namely $G_{ab} = 0$ or $R_{ab} = 0$ ($a, b = 0$–4) in terms of the 5D Einstein tensor or the

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Ricci tensor. These equations contain as a subset the 4D field equations with matter, namely $G_{\alpha\beta} = T_{\alpha\beta}$ (α, $\beta = 0$–3) in terms of the 4D Einstein tensor and an appropriately defined energy-momentum tensor (see ref. [6], p. 3885). The components of the latter for metric (1) are

$$\left\{\begin{aligned} T_0^0 &= \frac{1}{A^2}\left[\frac{\ddot{D}}{D} - \frac{\dot{A}\dot{D}}{AD} - \frac{\dot{A}' D'}{AD}\right] = \varrho\,,\\ T_1^1 &= -\frac{1}{A^2}\left[\frac{D''}{D} - \frac{\dot{A}\dot{D}}{AD} - \frac{A' D'}{AD}\right] = -p_1,\\ T_2^2 &= -\frac{1}{A^2}\left[-\frac{\dot{B}\dot{D}}{BD} + \frac{B' D'}{BD}\right] = -p_2,\\ T_3^3 &= -\frac{1}{A^2}\left[-\frac{\dot{C}\dot{D}}{CD} - \frac{C' D'}{CD}\right] = -p_3,\\ T_1^0 &= \frac{1}{A^2}\left[\frac{\dot{D}'}{D} - \frac{A'\dot{D}}{AD} - \frac{\dot{A} D'}{AD}\right] = Q\,. \end{aligned}\right. \tag{2}$$

Here an overdot denotes $\partial/\partial t$ and a prime denotes $\partial/\partial r$. We have introduced physical terminology for the density (ϱ), the pressure components in the 3 orthogonal directions of 3D space (p_1, p_2, p_3) and the heat flux (Q). In general, the 5D induced-matter approach leads to anisotropic pressure [6], but we will see that the effective 4D equations of state are radiation-like and physically acceptable.

The components of the Einstein tensor for metric (1) are

$$\left\{\begin{aligned} G_0^0 &= \frac{1}{A^2}\left[\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{A}\dot{D}}{AD} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{D}}{CD} + \frac{\dot{B}\dot{D}}{BD} + \frac{A'B'}{AB} + \frac{A'C'}{AC} + \frac{A'D'}{AD} - \right.\\ &\qquad\left. - \frac{B'C'}{BC} - \frac{B'D'}{BD} - \frac{C'D'}{CD} - \frac{B''}{B} - \frac{C''}{C} - \frac{D''}{D}\right],\\ G_1^1 &= \frac{1}{A^2}\left[\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\ddot{D}}{D} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{B}\dot{D}}{BD} + \frac{\dot{C}\dot{D}}{CD} - \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{A}\dot{C}}{AC} - \frac{\dot{A}\dot{D}}{AD} - \right.\\ &\qquad\left. - \frac{A'B'}{AB} - \frac{A'C'}{AC} - \frac{A'D'}{AD} - \frac{B'C'}{BC} - \frac{B'D'}{BD} - \frac{C'D'}{CD}\right],\\ G_2^2 &= \frac{1}{A^2}\left[\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\ddot{D}}{D} + \frac{\dot{C}\dot{D}}{CD} - \frac{\dot{A}^2}{A^2} + \frac{A'^2}{A^2} - \frac{A''}{A} - \frac{C''}{C} - \frac{D''}{D} - \frac{C'D'}{CD}\right],\\ G_3^3 &= \frac{1}{A^2}\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{D}}{D} + \frac{\dot{B}\dot{D}}{BD} - \frac{\dot{A}^2}{A^2} + \frac{A'^2}{A^2} - \frac{A''}{A} - \frac{B''}{B} - \frac{D''}{D} - \frac{B'D'}{BD}\right],\\ G_4^4 &= \frac{1}{A^2}\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\dot{A}^2}{A^2} + \frac{\dot{B}\dot{C}}{BC} + \frac{A'^2}{A^2} - \frac{A''}{A} - \frac{B''}{B} - \frac{C''}{C} - \frac{B'D'}{BC}\right],\\ G_1^0 &= \frac{1}{A^2}\left[\frac{A'\dot{B}}{AB} + \frac{A'\dot{C}}{AC} + \frac{A'\dot{D}}{AD} + \frac{\dot{A}B'}{AB} + \frac{\dot{A}C'}{AC} + \frac{\dot{A}D'}{AD} - \frac{\dot{B}'}{B} - \frac{\dot{C}'}{C} - \frac{\dot{D}'}{D}\right]. \end{aligned}\right. \tag{3}$$

We proceed to give 4 exact solutions of $G^a_b = 0$ using (3), together with their induced properties of matter as derived using (2). Their effective equations of state and singularity-indicating (5D) Kretschmann scalars K will also be given.

Case I:

$$
\left\{
\begin{aligned}
&A = \cosh(2mt)\cdot\cosh^4(mr)\,,\\
&B = m^{-1}\cosh(2mt)\cdot\cosh(mr)\cdot\sinh(mr)\,,\\
&C = \cosh(2mt)\cdot\cosh^2(mr)\,,\\
&D = [\cosh(2mt)]^{-1}[\cosh(mr)]^{-2},\\
&\varrho = 4m^2[\cosh(2mt)]^{-2}[\cosh(mr)]^{-8}[3\tanh^2(2mt)+2\tanh^2(mr)-1]\,,\\
&p_1 = 2m^2[\cosh(2mt)]^{-2}[\cosh(mr)]^{-8}[2\tanh^2(2mt)+7\tanh^2(mr)-1]\,,\\
&p_2 = 2m^2[\cosh(2mt)]^{-2}[\cosh(mr)]^{-8}[2\tanh^2(2mt)-\tanh^2(mr)-1]\,,\\
&p_3 = 4m^2[\cosh(2mt)]^{-2}[\cosh(mr)]^{-8}[\tanh^2(2mt)-\tanh^2(mr)]\,,\\
&Q = 16m^2\sinh(2mt)[\cosh(2mt)]^{-3}\sinh(mr)[\cosh(mr)]^{-9},\\
&\varrho = p_1+p_2+p_3\,,\\
&K = 194m^4[\cosh(mr)]^{-20}[\cosh(2mt)]^{-8}[8\cosh^2(2mt)\sinh^2(2mt)\cosh^4(mr)+\\
&\qquad\qquad +6\cosh^4(mr)-14\cosh^4(2mt)\cosh^2(mr)+9\cosh^4(2mt)]\,.
\end{aligned}
\right.
\tag{4}
$$

Case II:

$$
\left\{
\begin{aligned}
&A = \cosh(2mt)\cdot\cosh^4(mr)\,,\\
&B = m^{-1}\cosh(2mt)\cdot\cosh(mr)\cdot\sinh(mr)\,,\\
&C = [\cosh(2mt)]^{-1}[\cosh(mr)]^{-2}\,,\\
&D = \cosh(2mt)\cdot\cosh^2(mr)\,,\\
&\varrho = 4m^2[\cosh(2mt)]^{-2}[\cosh(mr)]^{-8}[\operatorname{sech}^2(2mt)-2\tanh^2(mr)]\,,\\
&p_1 = 2m^2[\cosh(2mt)]^{-2}[\cosh(mr)]^{-8}[1-2\tanh^2(2mt)-3\tanh^2(mr)]\,,\\
&p_2 = 2m^2[\cosh(2mt)]^{-2}[\cosh(mr)]^{-8}[1+\tanh^2(mr)-2\tanh^2(2mt)]\,,\\
&p_3 = 4m^2[\cosh(2mt)]^{-2}[\cosh(mr)]^{-8}[\tanh^2(2mt)-\tanh^2(mr)]\,,\\
&Q = 8m^2\sinh(2mt)[\cosh(2mt)]^{-3}\sinh(mr)[\cosh(mr)]^{-9},\\
&\varrho = p_1+p_2+p_3\,,\\
&K = 194m^4[\cosh(mr)]^{-20}[\cosh(2mt)]^{-8}[8\cosh^2(2mt)\sinh^2(2mt)\cosh^4(mr)+\\
&\qquad\qquad +6\cosh^4(mr)-14\cosh^4(2mt)\cosh^2(mr)+9\cosh^4(2mt)]\,.
\end{aligned}
\right.
\tag{5}
$$

Case III:

$$
\left\{
\begin{aligned}
&A = \cosh^2(mr)\,,\\
&B = m^{-1}[\cosh(mr)]^{-1}\sinh(mr)\,,\\
&C = 1\,,\\
&D = \cosh(2mt)\cdot\cosh^2(mr)\,,\\
&\varrho = 4m^2[\cosh(mr)]^{-6},\\
&p_1 = 2m^2[\cosh(mr)]^{-6},\\
&p_2 = 2m^2[\cosh(mr)]^{-6},\\
&p_3 = 0\,,\\
&Q = 0\,,\\
&\varrho = p_1 + p_2 + p_3,\\
&K = 192m^4[\cosh(mr)]^{-12}.
\end{aligned}
\right.
\tag{6}
$$

Case IV:

$$
\left\{
\begin{aligned}
&A = \cosh^2(mr)\,,\\
&B = m^{-1}[\cosh(mr)]^{-1}\sinh(mr)\,,\\
&C = \cosh(2mt)\cdot\cosh^2(mr)\,,\\
&D = 1\,,\\
&\varrho = 0\,,\\
&p_1 = p_2 = p_3 = 0\,,\\
&Q = 0\,,\\
&K = 192m^4[\cosh(mr)]^{-12}.
\end{aligned}
\right.
\tag{7}
$$

In the above, m is a constant. The solutions were derived analytically and confirmed by computer.

The preceding 4 solutions share some properties. For example, the effective 4D equation of state is that of radiation or highly relativistic particles, $\bar{p} \equiv \varrho/3$, where $\bar{p} \equiv (p_1 + p_2 + p_3)/3$ is the average 3D pressure. (Case IV above may be regarded as a limit case of this in which the pressure and density are both zero.) The fluids they describe extend to spatial infinity, so technically they are cosmological in nature. The density and pressure do not diverge at the origin of either space or time, so they are like the solutions studied by Banerjee *et al.* [5]. This is confirmed by their Kretschmann scalars, which are all finite at the origin of space and time, and in fact all equal to $192m^4$. (Cases I and II have K scalars that depend on t and r and are equal, while cases III and IV have K scalars that depend on r only and are equal.) Thus they are cylindrically symmetric cosmological models which, however, lack big-bang singularities.

The above results may be compared with some others for cosmological 5D metrics that have appeared in the literature [7, 13-15]. The simplest such metric is

$$ds^2 = dt^2 - t\, d\sigma^2 - t^{-1} d\psi^2, \tag{8}$$

where $d\sigma^2 \equiv dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ in spherical polars. Substituting this solution of the 5D field equations into the expression for the effective 4D energy-momentum tensor [6] gives the density, (isotropic) pressure and equation of state as

$$\varrho = \frac{3}{4t^2}, \qquad p = \frac{1}{4t^2}, \qquad \varrho = 3p. \tag{9}$$

The 5D Kretschmann scalar is

$$K = \frac{9}{2t^4}, \tag{10}$$

confirming that there is a big-bang singularity at $t = 0$. This behaviour is similar to that of the Kasner-like 5D solutions studied by Roque and Seiler [14]. They found several classes of solutions that generalize to the anisotropic case the isotropic 5D solutions of Ponce de Leon [13]. The solutions of Roque and Seiler [14] are Bianchi type I on hypersurfaces $\psi = \text{const}$, but in general the metric coefficients of the 3 spatial dimensions and the extra dimension can depend on t and ψ. The dependence on ψ in general leads to non-radiation equations of state for the induced-matter properties [6]. As for singularities, Roque and Seiler noted the existence of non-flat solutions whose Kretschmann scalars diverged for $t \to 0$ and $\psi \to 0$, and 5D flat solutions with zero K. They also found a non-flat solution with zero K (ref. [14], pp. 1157, 1165). The afore-mentioned 5D solutions of Ponce de Leon have flat space sections and reduce to 4D Friedmann-Robertson-Walker (FRW) ones on hypersurfaces $\psi = \text{const}$ [13]. His physically most interesting solution can be written:

$$ds^2 = \psi^2 dt^2 - t^{2/\alpha} \psi^{2/(1-\alpha)} d\sigma^2 - \alpha^2(1-\alpha)^{-2} t^2 d\psi^2, \tag{11}$$

where α is a parameter related to the induced 4D properties of matter:

$$\varrho = \frac{3}{\alpha^2\psi^2 t^2}, \qquad p = \frac{2\alpha - 3}{\alpha^2\psi^2 t^2}, \qquad p = \left(\frac{2\alpha}{3} - 1\right)\varrho. \tag{12}$$

The choice $\alpha = 2$ gives $p = \varrho/3$ and a scale factor that grows at $t^{1/2}$ (*i.e.* the radiation model for the early universe). The choice $\alpha = 3/2$ gives $p = 0$ and a scale factor that grows as $t^{2/3}$ (*i.e.* dust or the Einstein-de Sitter model for the late universe). The choice of coordinates in (11) clearly results in an excellent cosmology from the physical viewpoint. However, those coordinates obscure a property that is very remarkable from the mathematical viewpoint. Consider the coordinate transformation

$$
(13)\qquad \begin{cases} T = \left(\dfrac{\alpha}{2}\right) t^{1/\alpha} \psi^{1/(1-\alpha)} \left(1 + \dfrac{r^2}{\alpha^2}\right) - \dfrac{\alpha}{2(1-2\alpha)} [t^{-1} \psi^{\alpha/(1-\alpha)}]^{(1-2\alpha)/\alpha}, \\ R = r t^{1/\alpha} \psi^{1/(1-\alpha)}, \\ \Psi = \left(\dfrac{\alpha}{2}\right) t^{1/\alpha} \psi^{1/(1-\alpha)} \left(1 - \dfrac{r^2}{\alpha^2}\right) + \dfrac{\alpha}{2(1-2\alpha)} [t^{-1} \psi^{\alpha/(1-\alpha)}]^{(1-2\alpha)/\alpha}. \end{cases}
$$

Then (11) becomes

$$
(14)\qquad \mathrm{d}s^2 = \mathrm{d}T^2 - \mathrm{d}R^2 - R^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\psi^2) - \mathrm{d}\Psi^2,
$$

which is manifestly flat. (This may be confirmed by computer using (11) directly.) It should be noted that a coordinate transformation like (13) that involves the extra coordinate will preserve 5D geometrical invariants but not necessarily 4D properties of matter. Also, the 5D metric (11) has a 4D part which is not only an FRW model on a hypersurface $\psi = \text{const}$, but also in general curved (the 4D Ricci scalar is $6(\alpha - 2)/\alpha^2 t^2 \psi^2$). These comments lead to a rather interesting interpretation of the big-bang [15], namely that it can be regarded as the result of an unfortunate choice of coordinates in a truncated 5D Minkowski space.

3. – Conclusion

We have given 4 new cylindrically symmetric solutions of the “vacuum” equations in 5D and derived their induced 4D physical properties, eqs. (4)-(7). We have also commented on some known 5D cosmological solutions and their physical interpretation, eqs. (8)-(14). Our cylindrical solutions have density and pressure which do not diverge in a big-bang manner, but other cosmological solutions do show such behaviour. However, in one class of solutions of the latter type the divergence of the effective 4D properties of matter can be traced to a choice of 5D coordinates. We suggest further study of how 4D cosmological models can be embedded in 5D space, to help decide if the big-bang indicated by physical data is a mathematical artifact or not.

* * *

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